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BIOGRAPHY.

PROFESSOR FELIX KLEIN.

BY DR. GEORGE BRUCE HALSTED.

THE eminent subject of this very imperfect sketch was born on the twenty-fifth of April 1849 in Duesseldorf. His mother was Elise Sophie *nee* Kayser; his father, the "Landrentmeister" Caspar Klein, both of the protestant faith. For eight years, from the autumn of 1857 to the autumn of 1865 he attended the Duesseldorf Gymnasium, and went thence to the University of Bonn, for the study of mathematics and the natural sciences, especially physics. Here he had the extraordinary good fortune to come into close relations with the great Professor Pluecker, who gave him the position of assistant in the physical institute of Bonn, and used his help in writing out his profoundly original and stimulating mathematical works.

The death of Pluecker May 22nd 1868 closed this formative period, of which the influence on Klein can not be over estimated. So mighty is the power of contact with the living spirit of research, of taking part in original work with a master, of sharing in creative authorship, that any one who has once come intimately in contact with a producer of the first rank must have had his whole mentality altered for the rest of his life.

The gradual development, high attainment, and then continuous achievement of Felix Klein are more due to Pluecker than to all other influences combined. His very mental attitude in the world of mathematics constantly recalls his great maker.

Of others whose lectures he attended, we may mention Argelander and Lipschitz, to the latter of whom particularly he has expressed his gratitude for kindly and efficient guidance and aid in his studies. Klein took his doctor's

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degree at Bonn on December 12th 1868 with a dissertation "On the transformation of the general equation of the second degree between line-coordinates to a canonic form," a subject taken from the analytic line-geometry of his master Pluecker. A line-complex of the nth degree contains a triply infinite multitude of straights, which are so distributed in space, that those straights which go through a fixed point make a cone of the nth order, or, what is the same, that those straights which lie in a fixed plane envelop a curve of the nth class. Such an aggregate or form finds its analytic representation through the coordinates of the straight in space, introduced by Pluecker. According to Pluecker the straight has six homogeneous coordinates which fulfill an equation-of-condition of the second degree. By means of these the straight is determined with reference to a coordinate-tetrahedron. A homogeneous equation of the nth degree between these coordinates represents a complex of the nth degree.

The dissertation transforms the equation of the second degree between line-coordinates to a canonic form,in correspondence with a change of the coordinate-tetrahedron. It first gives the general formulas to be applied in such a transformation.

From these the problem appears algebraically as the simultaneous linear transformation of the complex to a canonic form, and of the equation-of-condition, which the line coordinates must fulfill, into itself. In carrying out these transformations, it attains to a classification of the complexes of the second degree into distinct species.

The dissertation is dedicated to Pluecker and contains eight specific references to Pluecker's "Neue Geometrie des Raumes, gegruendet auf die Betrachtung der geraden Linie als Raumelement." It is lucid and simple, but for depth and promise contrasts sharply with the great dissertation of Riemann, that "book with seven seals."

It may be interesting, as characteristic of this germinating state, to note that of his five theses the second calls attention to one of Cauchy's slips in logical rigor, slips now known to be so numerous that C. S. Peirce makes of them a paradox, maintaining that fruitfulness of Cauchy's work is essentially connected with its logical inaccuracy.

The third thesis declares the assumption of an ether unavoidable in the explanation of the phenomena of light.

The last thesis is the desirability of the introduction of newer methods in Geometry alongside the Euclidean in gymnasial teaching.

This serves, it seems, to emphasize my point that the long eight years of gymnasial so-called *training* left the seed still dormant, and only in Pleucker did it find the rain and the sun to call it to life and growth.

Within two years now the devolopment is amazing. Already in 1870 he is working with another great genius, Sophus Lie; and in 1871 is presented to the Goettingen Academy of Science his epoch-making paper, "Ueber die sogenannte Nicht-Euklidische Geometrie." Its aim is to present the mathematical results of the non-Euclidean geometry, in so far as they pertain to the

theory of parallels, in a new, intuitive way; its instrument is the mighty projective geometry, which he proves independent of all question of parallels. He perfects the projective metrics of Cayley by founding cross-ratio, after von Staudt, wholly without any use or idea of measurement. Then can be constructed a general projective expression for distance, related to an arbitrary surface of the second degree as Fundamental-surface (Cayley's Absolute). This projective metrics then gives, according to the species of Absolute used, a picture of the results of the parallel-theory in the space of Lobachwsky, of Euclid, of Riemann. But not merely a picture; they coincide to their innermost nature.

The paper begins by stating that, as well-known, the eleventh axiom of Euclid is equivalent to the theorem that the sum of the angles in a triangle equals two right angles. Legendre gave a proof that the angle-sum in a triangle cannot be greater than two right angles; but this proof, like the corresponding one in Lobachevsky, assumes the infinite length of the straight.

Drop this assumption, and the proof falls, else would it apply in surface spherics. Legendre showed furthar, that if in one triangle the angle-sum is two right angles, it is so in every triangle. We now know that this had been proven long before by Saccheri. But Professor Klein said that he heard the name of Saccheri for the first time in my address before the World's Science Congress. But it is claimed for Gauss that he was the first to distinctly state his conviction of the impossibility of proving the theorem of the equality of the angle-sum to two right angles. But it does not follow, as claimed by his Goettingen worshippers, that Gauss ever came to the conviction that a valid non-Euclidean geometry was possible until after it had been made simultaneously by John Bolyai and Lobachevsky, and perhaps long before by Wolfgang Bolyai. Certainly the world did not hear of it from Gauss. He published nothing on it.

In this non Euclidean geometry there appears a certain constant characteristic for the metrics of the space. By giving this an infinite value we obtain the ordinary Euclidean geometry. But if it has a finite value, we get a quite distinct geometry, in which, for example, the following theorems hold: The angle sum in a triangle is less than two right angles, and indeed so much the more so the greater the surface of the triangle. For a triangle whose vertices are infinitely separated, the angle-sum is zero. Through a point without a straight one can draw two parallels to the straight, that is, lines which cut the straight on the one or the other side in a point at infinity. The straights through the point which run between the two parallels no where cut the given straight. But on the other hand, in Riemann's marvellous inaugural lecture, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," is pointed out that the unboundedness of space, which is experiential, does not carry with it the infinity of space.

It is thinkable, and would not contradict our perceptional intuition, which always relates to a finite piece of space, that space is finite and comes back into itself.

The geometry of our space would then be like that of a tridimensional sphere in a four dimensional manifoldness. This representation carries with it that the angle-sum in a triangle, as in ordinary spherical triangles, is greater than two right angles, and indeed the more so, the greater the triangle The straight would then have no point at infinity, and through a given point no parallel to a given straight could be drawn. Now Cayley constructed his celebrated projective metrics to show how the ordinary Euclidean metrics may be taken as a special part of projective geometry. Klein generalizes Cayley and founds three metric geometries, the elliptic (Riemann's), the hyperbolic (Lobachevsky's), the parabolic, (Euclid's).

This little paper of 1871 contains the promise of much that is most genial in the after work of a man now generally considered as the most interesting and one of the very greatest of living mathematicians. Of all those splended and charming series of lectures with which Klein has made Goettingen so attractive to the whole world, the most delightful and epoch-making are those on non-Euclidean geometry, (Nicht-Euklidische Geometrie, I. Vorlesung, gehalten waehrend des Wintersemesters 1889-90 von F. Klein. Ausgearbeitet von Fr. Schilling. Zweiter Abdruck. Goettingen 1893. Small Quarto, lithographed, pp. v. 365. II. Sommersemesters 1890. Zweiter Abdruck 1893. pp. iv. 238.)

The World's Science Congress at Chicago was in nothing more fortunate than in the presence of Helmholtz and Felix Klein, and in the spontaneous and universal homage accorded them no idea was more often emphasized than their connection with the birth and development of that wonderful new world of pure science typified in the non-Euclidean geometry.

The narrow limits of this feeble sketch prevent the statement of how much promise, richly fulfilled in the development of this many-sided man, in totally other directions is contained in a little-known paper of 1873, "Ueber den allgemeinen Functionsbegriff und dessen Darstellung durch eine willkuerliche Curve."

Twenty years of production and achievement have not in the least dampened the ardour of this enthusiastic mind. This very summer at the great meeting of scientists in Vienna Klein seemed the busyest, the foremost of all that goodly company.

ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By Professor P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana.

[Continued from the November Number.]

PROPOSITION VII. If in a quadrilateral two sides are equal, and the